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PATCHWORKING SINGULAR ALGEBRAIC CURVES I

BY

EUGENII SHUSTIN*

School of Mathematical Sciences, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
e-mail: shustin@post.tau.ac.il

AND

ILYA TYOMKIN*

School of Mathematical Sciences, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
and
Institut de Mathématiques, Algèbres d'opérateurs et répresentations
Université Pierre et Marie Curie
175 rue du Chevaleret, plateu 7D, F-75013 Paris, France
e-mail: tyomkin@post.tau.ac.il

ABSTRACT

In this paper we present a general patchworking procedure for the construction of reduced singular curves having prescribed singularities and belonging to a given linear system on algebraic surfaces. It originates in the Viro "gluing" method for the construction of real non-singular algebraic hypersurfaces. The general procedure includes almost all known particular modifications, and goes far beyond. Some applications and examples illustrate the construction.

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1. Introduction

In this paper we develop the patchworking method for constructing algebraic curves with prescribed singularities in a given linear system on an algebraic surface.

Our procedure originates in the Viro "gluing" method [22, 23, 24, 25, 26, 27] invented in 1979–80 for the construction of real algebraic non-singular hypersurfaces with prescribed topology, and which provided a major breakthrough in Hilbert's 16th problem [8]. Later, it was modified for the construction of algebraic curves with many prescribed singularities in the plane [14, 16] or on other algebraic surfaces [11, 18], and, more generally, hypersurfaces with prescribed singularities in smooth algebraic varieties [17], construction of polynomials with prescribed critical points [15, 16], vector fields with many limit cycles and prescribed singularities [9], and some other problems, for example, enumeration of singular curves [13, 19]. We should like also to mention that the patchworking construction appears to be useful in the symplectic setting as well [4, 10].

In general, the patchworking construction starts with a one-dimensional flat family $X \to (\mathbb{C}, 0)$ of algebraic surfaces with an irreducible connected general fibre X_t, $t \neq 0$, and reduced reducible central fibre X_0, in which an algebraic curve C_0 is given. The construction produces a flat family $C_t \subset X_t$, $t \in (\mathbb{C}, 0)$, of curves such that the curves C_t, $t \neq 0$, inherit some properties, say, singularity types, from the components of the given central curve C_0. Topologically, a curve C_t, $t \neq 0$, is glued up out of the components of C_0 sitting in distinct components of X_0.

In the present paper we focus on the construction of families of curves C_t, which induce an equisingular (or equianalytic) deformation for the singular points of C_0. This requirement can be reduced to the condition of the smoothness and transversality of certain equisingular (or equianalytic) strata, which in turn can be expressed in the form of H^1-vanishing conditions for ideal sheaves of certain zero-dimensional schemes in X_0.

In the two examples of patchworking treated in [14, 16, 17] (and which we present in more detail in section 2.1) it was supposed that the components of the curve C_0 cross the intersection lines of the components of X_0 at their non-singular points and transversally. In [2, 3] a version of the patchworking procedure has been used for the construction of nodal curves of surfaces in $\mathbb{P}^3$. In the latter situation, the components of C_0 were tangent up to some order to the intersection lines of the components of X_0. Furthermore, in [2, 3] the deformation of such a singular point of C_0 is treated as a specific deformation of a planar curve singularity of type A_k, and this is done by means of [1], which is rather technical. Our idea (which is developed in detail in [20]) is to reduce this problem to the transversal case, and then to use the patchworking procedure, Theorems 2.8 and 2.15. Namely we blow up such a singular point of C_0 and add a new component E to X_0 so that the deformation of the (boundary) singular point of C_0 is represented as a patchworking of a singular curve in E. This approach generalizes, in fact, to arbitrary singularities of C_0 along the intersection lines of the components of X_0. We point out that in the general situation, these singularities of C_0 are no longer planar ones. Another novelty of our approach (which is also presented in [20]) is that we can describe in a controlled way deformations of curves C_0 having multiple components. Again, the idea is to blow up along the multiple components and, adding new components to X_0 and C_0, represent this deformation as a patchworking of a curve with isolated singularities. We should mention that the purpose of the current paper is to develop the patchworking techniques in the most general setting. And most of the examples and applications are contained in the second part of the paper [20].

This paper is organized as follows: in section 2 we formulate the general patchworking procedure in Theorem 2.8 (weak version) and in Theorem 2.15 (strong version). In section 3 we prove Theorem 3.1, whose purpose is to give a method for verifying the condition of the weak patchworking theorem. Finally, section 4 presents two examples illustrating the patchworking procedure. One of the examples is a generalization of the result of Chiantini and Ciliberto [3] to the case of arbitrary simple singularities, and another example illustrates the advantage of the strong patchworking theorem over the weak version.

We consider further examples and applications in [20]. It includes the following topics: new bounds for the existence of singular curves with prescribed singularities on algebraic surfaces, description of a natural space of deformations of curves with non-isolated singularities, and an extension of the patchworking procedure to the case of non-transversal boundary conditions.

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2. The geometric patchworking

2.1 TWO EXAMPLES. Let us explain the idea of the patchworking. Assume that we are given a surface Σ and a finite set P of singularity types. Consider a partition of P: $P = \bigcup P^i$. Assume in addition that we can degenerate our surface Σ into a union of some other surfaces Σ^i and on each Σ^i we have a curve C^i having singularities exactly of types $\mathcal{S} \in P^i$. Then, under appropriate conditions, one can "glue" these curves into a curve on Σ preserving the singularities. We can reformulate it as follows: We start with the following initial data.

- A family of projective surfaces π: $X \to T$, with an irreducible connected general fiber X_t and a reducible closed fiber $X_0 = \bigcup \Sigma^i$,
- A family of line bundles $\mathcal{L}_t$ on X_t, or equivalently a line bundle $\mathcal{L}$ on X, and
- A singular curve $C_0 = \bigcup C_0^i \subset \bigcup \Sigma^i$ (with given types of singularities) defined by some section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$.

Then, if the initial patchworking data satisfy some properties (see the next section for explicit definitions) we can construct a deformation $C_t \subset X_t$ of C_0 preserving the singularities of C_0. Let us present two typical examples of the patchworking data:

Example 2.1: First we define the family of surfaces. Consider $\widehat{\pi}$: $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $z_1, \dots, z_k \in \widehat{\pi}^{-1}(0) \cong \mathbb{P}^2$ be distinct points. Define X to be the blow up of $\mathbb{P}^2 \times \mathbb{P}^1$ along $z_1, \dots, z_k \in \widehat{\pi}^{-1}(0)$. Thus our family is given by π: $X \to \mathbb{P}^1$. Now we shall define the family of line bundles. Consider

$$\mathcal{L} = \pi^*_{\mathbb{P}^2}\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_X(-m_1E_1) \otimes \cdots \otimes \mathcal{O}_X(-m_kE_k),$$

where $\pi_{\mathbb{P}^2}$: $X \to \mathbb{P}^2$ is the natural projection, and $d, m_1, \dots, m_k \in \mathbb{N}$. Last we have to present the section. On each exceptional plane Σ^i we choose a curve C_0^i of degree m_i having exactly one singular point of some type $\mathcal{S}^i$ and intersecting $\Sigma^i \cap \Sigma^0$ transversally at m_i different points $p_1^i, \dots, p_{m_i}^i$. On the blown up plane Σ^0 we choose a curve C_0^0 of type $dL - \sum_{i=1}^k m_i(\Sigma^i \cap \Sigma^0)$ such that $C_0^0 \cap \Sigma^i = C_0^i \cap \Sigma^0 = \bigcup_{j=1}^{m_i} p_j^i$. We choose a section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$ to be any section defining the curve $C_0 = \bigcup C_0^i$ (any two such sections differ only by a multiplicative scalar).

The next example represents the original Viro's gluing procedure [27].

Example 2.2: Consider the triangle $\Delta = \{(0,0),(d,0),(0,d)\} \subset \mathbb{R}^2$ and its subdivision $\Delta = \bigcup_{i=1}^k \Delta^i$ into a union of convex polygons with integral vertices. We assume that the subdivision is convex, i.e., there exists a convex continuous piecewise linear function $\nu\colon \Delta \to \mathbb{R}$ whose linearity domains are exactly Δ^i, $i = 1, \dots, k$, and $\nu(\mathbb{Z}^2 \cap \Delta) \subseteq \mathbb{Z}_-$. Consider the convex hull Ξ of the polygon Δ and the graph $\mathrm{Graph}(\nu)$ in $\mathbb{R}^3$. We define X to be the toric threefold corresponding to Ξ.

Adding the undergraph of ν to Ξ, we obtain the polyhedron $\hat{\Xi} = \Delta \times (-\infty, 0]$. The corresponding toric threefold $\hat{X} = \mathbb{P}^2 \times (\mathbb{P}^1 \backslash \{\text{point}\})$ embeds into X as the complement to $X_0 \simeq \bigcup_{i=1}^k \mathrm{Tor}(\Delta^i) = \bigcup_{i=1}^k \Sigma^i$, the union of the divisors, determined by the faces of $\mathrm{Graph}(\nu)$. The projection $\hat{X} \to \mathbb{P}^1 \backslash \{\text{point}\}$ extends up to a projection $X \to T = \mathbb{P}^1$ with the central fibre X_0 and other fibres isomorphic to $\mathbb{P}^2$. We define $\mathcal{L} = \mathcal{L}(\Xi)$ to be the tautological line bundle of the toric variety $X = \mathrm{Tor}(\Xi)$. Then $\mathcal{L}_{|X_t} \cong \mathcal{O}_{\mathbb{P}^2}(d)$ for $t \neq 0$ and $\mathcal{L}_{|\Sigma^i}$ is the tautological bundle of Σ^i. The last ingredient of the patchworking data to be chosen is a section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$. We refer to [22], [27], [24] and [16] for concrete examples.

2.2 The Patchworking Data.

Convention 2.3: *In this paper we work exclusively over $\mathbb{C}$, which, by the Lefschetz principle, can be replaced by any algebraically closed field of characteristic zero.*

Consider the following data:

- A one-parameter flat family of projective surfaces $\pi\colon X \to T$ over a smooth base T,
- A family of invertible sheaves $\mathcal{L}_t$ on $X_t = \pi^{-1}(t)$, i.e., an invertible sheaf $\mathcal{L}$ on X (up to a twist by $\pi^*\mathcal{F}$ where $\mathcal{F}$ is a line bundle on T), and
- A section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$ (the zero set of ξ_0 is exactly the set of curves we are going to glue up).

Assume that our data satisfies the following properties

$X1$. X_t is reduced and irreducible for any $t \neq 0$, where $0 \in T$ is a distinguished point.

$X2$. $X_0 = \bigcup_{i=1}^k \Sigma^i$ is a union of reduced and irreducible surfaces such that $\dim(\Sigma^i \cap \Sigma^j \cap \Sigma^k) = 0$, for any three distinct indices i, j, k.

Notation 2.4: We denote the zero set of ξ_0 by C_0. Then $C_0 = \bigcup_{i=1}^k C_0^i$, where $C_0^i = C_0 \cap \Sigma^i$.

$S1$. C_0 has only isolated singular points and all these points are smooth points of X. If $p \in \mathrm{Sing}(X_0) \cap \mathrm{Sing}(C_0)$ then $p \in \Sigma^i \cap \Sigma^j$ for some i, j.

$S2$. C_0^i are reduced.

$S3$. $C_0 \cap \Sigma^i \cap \Sigma^j$ is reduced for any $i \neq j$.

$S4$. $C_0 \cap \Sigma^i \cap \Sigma^j \cap \Sigma^k = \emptyset$, for any three distinct indices i, j, k.

$S5$. For any $p \in C_0 \cap \Sigma^i \cap \Sigma^j$, there exists an open analytic neighborhood $p \in U \subset X$ such that $X_0 \cap U \subset U$ is a **quasi-normal crossing divisor**; i.e., either it is a normal crossing divisor or the pair $U \to U_\epsilon(0) \subset T$ is isomorphic to an open analytic neighborhood of 0 of the pair

$$\operatorname{Spec} \mathbb{C}[x, y, z, t]/(xy - t^l) \to \operatorname{Spec} \mathbb{C}[t]$$

for some positive integer l.

Remark 2.5: It is important that the last condition is stable under base change since, clearly, any one-dimensional base change replaces the exponent l of t in the above formula by another exponent.

It is easy to see that the two examples we considered in the introduction satisfy all the properties $X1, X2, S1 - S5$. To formulate the patchworking theorem we need several notations

Notation 2.6: Let $n(i)$ be the number of the singular points of C_0^i. We denote these singular points by $p_1^i, \ldots, p_{n(i)}^i$ and their singularity types by $\mathcal{S}_j^i = \mathcal{S}(p_j^i)$, $j = 1, \ldots, n(i)$. The types $\mathcal{S}_j^i$ can coincide for different j.

Notation 2.7: Let $\mathcal{I}_{p_j^i} \subset \mathcal{O}_{X_0, p_j^i}$ be the equisingular/equianalytic ideal of the singular point p_j^i. We denote by $\mathcal{I}$ the equisingular/equianalytic ideal sheaf of the zero-dimensional scheme Z, concentrated at $\bigcup_{i,j} p_j^i \subset X_0$, which is defined locally at p_j^i by the ideal $\mathcal{I}_{p_j^i}$ (for the precise definition of the equisingular/equianalytic ideal we refer to [6], Section 1).

2.3 Weak Patchworking Theorem. We start with the patchworking data (and notations) from the previous section, namely, we are given:

- a family of surfaces $\pi\colon X \to T$,
- a line bundle $\mathcal{L}$,
- a section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$,

satisfying all the properties $X1$, $X2$, $S1$, $S2$, $S3$, $S4$, $S5$ from the previous section. Now we can formulate the main result

Theorem 2.8 (Weak Patchworking Theorem): *Assume that*

$$H^1(X_0, \mathcal{I} \otimes \mathcal{L}_0) = 0. \tag{1}$$

Then there exists some open neighborhood $U_\epsilon = U_\epsilon(0) \subset T$ and a family of curves $C_t \in |\mathcal{L}_t|$, $t \in U_\epsilon$, having $\sum_i n(i)$ singular points of types $\{\mathcal{S}_j^i\}_{1 \le j \le n(i)}^{1 \le i \le k}$, respectively, as their only singularities.

Remark 2.9: (i) T is a smooth curve and π is projective, hence $\pi_*\mathcal{L}$ is a direct sum of a locally free coherent sheaf with a torsion sheaf, but π is flat and $\mathcal{L}$ is invertible, thus the torsion part of $\pi_*\mathcal{L}$ is zero. Hence $\pi_*\mathcal{L}$ is a vector bundle on T.

(ii) Condition (1) implies $H^1(X_0, \mathcal{L}_0) = 0$. Hence for any $t \in T$ close to $0 \in T$, the canonical map $\pi_*\mathcal{L} \otimes k(t) \to H^0(X_t, \mathcal{L}_t)$ is an isomorphism.

Remark 2.10: Twisting $\mathcal{L}$ by $\pi^*\mathcal{F}$, where $\mathcal{F}$ is a very big line bundle, we can assume that the global sections of $\pi_*(\mathcal{L})$ generate its fibre at any point $t \in T$. Such a twist does not change the family of line bundles on the fibres, so we can assume that

$$\mathrm{res}_0 : H^0(X, \mathcal{L}) \to H^0(X_0, \mathcal{L}_0)$$

is surjective. Moreover, due to the previous remark, we can choose a subspace $H \subset H^0(X, \mathcal{L})$ such that the restriction map

$$\mathrm{res}_t : H \to H^0(X_t, \mathcal{L}_t)$$

is an isomorphism for all t in some open neighborhood $U_\epsilon(0)$ of 0. From now on we will identify H with $H^0(X_t, \mathcal{L}_t)$ for small t.

2.4 The Proof of Theorem 2.8.

Notation 2.11: Let M_0 denote the germ at ξ_0 of the equisingular/equianalytic family of sections $\alpha \in H^0(X_0, \mathcal{L}_0)$ having exactly $\sum_i n(i)$ singular points of topological/analytic types $\{\mathcal{S}_j^i\}_{1\le i\le k}^{1\le j\le n(i)}$.

To prove Theorem 2.8 we will need the following

Lemma 2.12 (Main Lemma): *Assume that*

1. M_0 *is smooth,*
2. $\mathrm{codim}(M_0 \subseteq H^0(X_0, \mathcal{L}_0))$ *is as expected (i.e., equal to the sum of the codimensions of the equisingular/equianalytic strata of the singularity types* $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k}$*).*

Then there exists some open neighborhood $U_\epsilon = U_\epsilon(0) \subset T$ *and a family of curves* $C_t \in |\mathcal{L}_t|$, $t \in U_\epsilon$, *having* $\sum_i n(i)$ *singular points of types* $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k}$, *respectively, as their only singularities.*

Proof:

STEP 1: First, we construct a deformation $M_t \subset H \times \{t\}$, $t \in U_\epsilon(0)$ of $M_0 \subset H \times \{0\}$. Let P_j^i be a singular point of C_0^i. Consider a small analytic neighborhood $V_\epsilon(P_j^i) \times U_\epsilon(0) \subset X$ of P_j^i, $V_\epsilon(P_j^i) \subset X_0$ and $U_\epsilon(0) \subset T$, and introduce local analytic coordinates x, y, t centered at P_j^i. We can assume that $\mathcal{L}|_{V_\epsilon(P_j^i)\times U_\epsilon(0)}$ is trivial. Thus $\mathrm{res}^X_{V_\epsilon(P_j^i)\times\{t\}}(H) = \mathrm{res}^{X_t}_{V_\epsilon(P_j^i)\times\{t\}}(H^0(X_t, \mathcal{L}_t)) \subset \mathbb{C}[[x,y]]$ provides us with a family of subspaces of fixed (finite!) dimension. Consider $ESF_j^i \subset \mathbb{C}[[x,y]]$ the germ of equisingular/equianalytic stratum of $\mathcal{S}(P_j^i)$. Then

$$M_0 = \bigcap_{i,j} (\mathrm{res}^{X_0}_{V_\epsilon(P_j^i)\times\{0\}})^{-1}(ESF_j^i) \subset H^0(X_0, \mathcal{L}_0) \cong H.$$

We define M_t to be $\bigcap_{i,j} (\mathrm{res}^{X_t}_{V_\epsilon(P_j^i)\times\{t\}})^{-1}(ESF_j^i) \subset H^0(X_t, \mathcal{L}_t) \cong H$.

STEP 2: Second, we prove that $\dim M_t = \dim M_0$. We know that M_0 is smooth and $\mathrm{codim}(M_0)$ is expected. This implies:

- The intersection $ESF_j^i \cap \mathrm{res}^X_{V_\epsilon(P_j^i)\times\{0\}}(H)$ is transversal. Hence the intersection $ESF_j^i \cap \mathrm{res}^X_{V_\epsilon(P_j^i)\times\{t\}}(H)$ is also transversal.
- The intersection $\bigcap_{i,j} (\mathrm{res}^{X_0}_{V_\epsilon(P_j^i)\times\{0\}})^{-1}(ESF_j^i) \subset H^0(X_0, \mathcal{L}_0)$ is transversal. Hence, for small t, the intersection $\bigcap_{i,j} (\mathrm{res}^{X_t}_{V_\epsilon(P_j^i)\times\{t\}})^{-1}(ESF_j^i) \subset H^0(X_t, \mathcal{L}_t)$ is transversal as well.

It follows now that $\dim M_t = \dim M_0 \geq 0$. Hence there exists a deformation $C_t \in |\mathcal{L}_t|$ of $C_0 \in |\mathcal{L}_0|$, such that for any t small enough the curve C_t has singular points of types $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k}$.

STEP 3: Last, we show that the curve C_t has no other singularities except $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k}$. Consider sufficiently fine open covering

$$C_0 \subset \bigcup_{ij} V_j^i \cup \bigcup_{ij\alpha} W_\alpha^{ij} \cup \bigcup_{\beta=1}^{r} U_\beta \subset X,$$

where V_j^i is a small neighborhood of the singular point of type P_j^i, W_α^{ij} are small neighborhoods of the points $z_\alpha \in C_0 \cap \Sigma^i \cap \Sigma^j$ and U_β does not intersect some small neighborhood of singular points of C_0. Choose a small equisingular deformation $C_t \in M_t$ of C_0 as above. Then, for small t,

$$C_t \subset \bigcup_{ij} V_j^i \cup \bigcup_{ij\alpha} W_\alpha^{ij} \cup \bigcup_{\beta=1}^{r} U_\beta.$$

It is clear that if t is small, then $C_t \cap U_\beta$ is smooth for any β. Let us show that $C_t \cap W^{ij}_\alpha$ is also smooth. If W^{ij}_α is small enough, we can choose local coordinates x, y, z in W^{ij}_α in such a way that our family of surfaces is given by $xy = t^l$ ($X_0 \subset X$ is a quasi-normal crossing divisor in the neighborhood of z_α) and the family C_t is given by $z + t^l \cdot f(x,y,z,t) = 0$ for some $f \in \mathbb{C}[[x,y,z,t]]$ (this follows from the condition $S3$). Now it is obvious that $C_t \cap W^{ij}_\alpha$ is smooth for $t \neq 0$ if W^{ij}_α is sufficiently small. To show that $C_t \cap V^i_j$ has only one singular point we will use the following well-known claim.

Claim 2.13: *Let $V \subset \mathbb{C}^2$ be a small disc, and let $C_0 \subset V$ be a curve with isolated singular point at the origin and no other singularities. Consider a small deformation C_t of C_0. Then*

$$\mu_V(C_t) \leq \mu_V(C_0),$$

where $\mu_V(C)$ is the sum of Milnor numbers of singular points of C in V.

Now we can finish the proof. We can choose V^i_j in such a way that $V^i_j \to U_\delta(0) \subset T$ is a trivial fibration (i.e., $V^i_j \cong V \times U_\delta(0)$). C_t has a singular point in $V^i_j \cap X_t \cong V$ of the same type as C_0, so it follows from Claim 2.13 that C_t has no other singularities in V^i_j. ∎

Now Theorem 2.8 follows from the Main Lemma. Indeed, along the general deformation theory, the cohomology vanishing hypothesis (1) of Theorem 2.8 implies the unobstructedness of the equisingular/equianalytic deformations together with the smoothness and transversality of intersection of the corresponding equisingular/equianalytic strata mentioned in Main Lemma (cf. [5], Proposition 3.7 and Theorem 6.1(ii), and [6], Theorem 3.6(b), where the assumption on the smoothness of the ambient surface can be replaced by the assumption that the curve singular points, which are traced in the deformation, lie outside the singular locus of the surface). ∎

2.5 Strong Patchworking Theorem. In this subsection we would like to present a stronger version of the patchworking procedure which turns out to be very useful in some applications.

It is easy to see that for all $t \neq 0$ small enough, the natural map $\pi_* \mathcal{L} \otimes k(t) \to H^0(X_t, \mathcal{L}_t)$ is an isomorphism, and for $t = 0$ this map is injective. Assume that the curve C_0 is given by a section $\xi_0 \in \operatorname{Im}(\pi_* \mathcal{L} \otimes k(0)) \subseteq H^0(X_0, \mathcal{L}_0)$. Modifying Notation 2.11 we define N_0 to be the germ at C_0 of equisingular family of sections $\alpha \in \operatorname{Im}(\pi_* \mathcal{L} \otimes k(0))$ having exactly $\sum_i n(i)$ singular points of topological (analytic) types $\{\mathcal{S}_j^i\}_{1\le i\le k}^{1\le j\le n(i)}$. Now replacing M_0 by N_0 and repeating word by word the proof of the Main Lemma one can prove the following

LEMMA 2.14: *If*

1. N_0 *is smooth,*
2. $\operatorname{codim}(N_0 \subseteq \operatorname{Im}(\pi_*\mathcal{L}\otimes k(0)))$ *is expected.,*

then there exists some open neighborhood $U_\epsilon = U_\epsilon(0) \subset T$ *and a family of curves* $C_t \in |\mathcal{L}_t|$, $t \in U_\epsilon$, *having* $\sum_i n(i)$ *singular points of types* $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k}$, *respectively, as their only singularities.*

Now we can present the Strong Patchworking Theorem. As before, we start with the patchworking data (and notations) from Section 2.2, namely, we are given

- a family of surfaces $\pi: X \to T$,
- a line bundle $\mathcal{L}$,
- a section $\xi_0 \in H^0(X_0, \mathcal{L}_0)$,

satisfying all the properties from Section 2.2, namely $X1$, $X2$, $S1$, $S2$, $S3$, $S4$, $S5$.

THEOREM 2.15 (Strong Patchworking Theorem): *Assume that*

$T1$. $\xi_0 \in \operatorname{Im}(\pi_*\mathcal{L}\otimes k(0))$,

$T2$. *the natural map* $H^1(X_0, \mathcal{I}\otimes\mathcal{L}_0) \to H^1(X_0, \mathcal{L}_0)$ *is an isomorphism, and*

$T3$. *the intersection* $H^0(X_0, \mathcal{I}\otimes\mathcal{L}_0) \cap \operatorname{Im}(\pi_*\mathcal{L}\otimes k(0)) \subset H^0(X_0, \mathcal{L}_0)$ *is transversal.*

Then there exists some open neighborhood $U_\epsilon = U_\epsilon(0) \subset T$ *and a family of curves* $C_t \in |\mathcal{L}_t|$, $t \in U_\epsilon$, *having* $\sum_i n(i)$ *singular points of types* $\{\mathcal{S}_j^i\}_{1\le j\le n(i)}^{1\le i\le k}$, *respectively, as their only singularities.*

Proof: Consider the exact sequence

$$H^0(X_0, \mathcal{L}_0) \to H^0(Z, \mathcal{O}_Z) \to H^1(X_0, \mathcal{I}\otimes\mathcal{L}_0) \to H^1(X_0, \mathcal{L}_0),$$

where the zero-dimensional scheme $Z \subset X_0$ is defined in 2.7. Then condition $T2$ implies the surjectivity of $H^0(X_0, \mathcal{L}_0) \to H^0(Z, \mathcal{O}_Z)$, the latter space being $T^1_{C_0, \operatorname{Sing}(C_0)\setminus \operatorname{Sing}(X_0)}$, and hence we get the smoothness of the germ of the equisingular deformation of C_0 (cf. [5], section 2.3). Moreover, it follows that the codimension of this germ is the expected one. Conditions $T1$ and $T3$ imply that the intersection of the germ of the equisingular deformation of ξ_0 with $\operatorname{Im}(\pi_*\mathcal{L}\otimes k(0))$ is non-empty and transversal in $H^0(X_0, \mathcal{L}_0)$. Hence this intersection is smooth and has expected dimension as well. Now we can apply Lemma 2.14, and the result follows. ∎

3. A useful theorem

Next we will give sufficient conditions for (1). Consider the set $V = \{\Sigma^i\}_{i=1}^k$ and the set $E = \{e_{x,y} = x \cap y | x, y \in V, \dim(x \cap y) = 1\}$. Let Γ be any connected oriented graph without cycles for which the set of vertices $V(\Gamma)$ is equal to V and the set of edges $E(\Gamma)$ is equal to E. For any $1 \le i \le k$, we introduce

- the set E^i of the intersection lines $e = \Sigma^i \cap \Sigma^j$, $j \neq i$, corresponding to the edges of Γ directed out of Σ^i,
- the line bundle

$$\mathcal{L}_\Gamma^i = \mathcal{L}_{|_{\Sigma^i}} \otimes \bigotimes_{e \in E^i} \mathcal{O}_{\Sigma^i}(-e).$$

THEOREM 3.1: *Assume that, for any $i = 1, \dots, k$,*

$$H^1(\Sigma^i, \mathcal{I}_{|_{\Sigma^i}} \otimes \mathcal{L}_\Gamma^i) = 0. \tag{2}$$

Then (1) holds true.

Remark 3.2: Besides keeping prescribed singularities in the patchworking construction, one can also preserve the tangency to certain curves. Namely, in the above notation, let $C' \to T$, $C' \subset X$, be a flat family of curves such that $C'_0 \cap C_0$ is finite and non-empty. Let $K \subset C'_0 \cap C_0 \backslash (\mathrm{Sing}(C_0) \cup \mathrm{Sing}(C'_0) \cup \mathrm{Sing}(\Sigma_0))$, $K \neq \emptyset$. Then we impose the requirement that each point $z \in K$ extends up to a section $z(t) \in C'_t$, $z(0) = z$, $t \in T$, such that the intersection number of C_t and C'_t persists:

$$(C_t \cdot C'_t)_{z(t)} = (C_0 \cdot C'_0)_z, \quad t \in T.$$

To satisfy this requirement we modify conditions (1), (2) by replacing $\mathcal{I}$ with $\mathcal{I} \otimes \mathcal{I}'$, where $\mathcal{I}'$ is the ideal sheaf of the zero-dimensional scheme concentrated at K and defined by ideals

$$I_z = \{\varphi \in \mathcal{O}_{\Sigma_0, z} : (\varphi \cdot C'_0)_z \ge (C_0 \cdot C'_0)_z - 1\}, \quad z \in K.$$

3.1 PROOF OF THEOREM 3.1. To prove the implication (2) $\Longrightarrow$ (1) we shall use the following

LEMMA 3.3: *Assume that we are given a reducible algebraic surface $Y = W \cup Z$ such that $D = W \cap Z$ is a divisor on both W and Z. Let $\mathcal{F}$ be a quasicoherent sheaf on Y satisfying*

$$H^1(W, \mathcal{F}_{|_W}) = H^1(Z, \mathcal{F}_{|_Z} \otimes \mathcal{O}_Z(-D)) = 0.$$

Then $H^1(Y,\mathcal{F})=0$.

Proof: Consider the exact sequence of sheaves on Y

$$0\to\pi_*^Z(\mathcal{F}_{|Z}\otimes\mathcal{O}_Z(-D))\to\mathcal{F}\to\pi_*^W(\mathcal{F}_{|W})\to 0$$

and its exact cohomology sequence

$$\begin{array}{ccccc}
H^1(Y,\pi_*^Z(\mathcal{F}_{|Z}\otimes\mathcal{O}_Z(-D))) & \longrightarrow & H^1(Y,\mathcal{F}) & \longrightarrow & H^1(Y,\pi_*^W(\mathcal{F}_{|W}))\\
\| & & & & \|\\
H^1(Z,\mathcal{F}_{|Z}\otimes\mathcal{O}_Z(-D)) & & & & H^1(W,\mathcal{F}_{|W})\\
\| & & & & \|\\
0 & & & & 0
\end{array}$$

From the exact sequence we see that $H^1(X,\mathcal{F})=0$. ∎

Next we notice that the graph Γ defines a partial ordering on the set of components of X_0. We can complete this ordering to a linear order. So we can assume that, for any $e_{ij}\in E^i$, the inequality $j>i$ is satisfied. We will use the following

Notation 3.4: For any $1\le m\le k$, the surface $\bigcup_{i=1}^m\Sigma^i\subset X_0$ will be denoted X_0^m, and the quasicoherent sheaf

$$(\mathcal{I}\otimes\mathcal{L})_{|X_0^m}\otimes\mathcal{O}_{X_0^m}\Big(-\sum_{i\le m}\sum_{j>m}\Sigma^i\cap\Sigma^j\Big)$$

on X_0^m will be denoted $\mathcal{F}^m$.

Now we proceed by induction, proving that $H^1(X_0^m,\mathcal{F}^m)=0$ for any $1\le m\le k$. If $m=1$ there is nothing to prove, since $H^1(X_0^1,\mathcal{F}^1)=H^1(\Sigma^1,\mathcal{I}\otimes\mathcal{L}_\Gamma^1)=0$. Assume that we proved $H^1(X_0^m,\mathcal{F}^m)=0$ for some $1\le m<k$. Let us prove it for $m+1$. $X_0^{m+1}=X_0^m\cup\Sigma^{m+1}$, so applying Lemma 3.3 for $Y=X_0^{m+1},Z=X_0^m,W=\Sigma^{m+1}$ it is enough to prove that

- $H^1(X_0^m,\mathcal{F}^{m+1}_{|X_0^m}\otimes\mathcal{O}_{X_0^m}(-X_0^m\cap\Sigma^{m+1}))=0$,
- $H^1(\Sigma^{m+1},\mathcal{F}^{m+1}_{|\Sigma^{m+1}})=0$.

But the first equality is just the induction assumption. And the second equality is given since $\mathcal{F}^{m+1}_{|\Sigma^{m+1}}=\mathcal{I}\otimes\mathcal{L}_\Gamma^{m+1}$. So $H^1(X_0,\mathcal{I}\otimes\mathcal{L}_0)=H^1(X_0^k,\mathcal{F}^k)=0$. This completes the proof of the theorem. ∎

4. Two examples

In this section we would like to present two examples (Theorem 4.2 and Theorem 4.5). The first example is an application of the patchworking, which generalizes the result of Chiantini and Ciliberto [3]. The second example illustrates a case where the strong version of patchworking could be applied and the weak version could not. We should mention that Theorem 4.5 was first proven in [17] by tedious direct computations.

4.1 The First Example.

Definition 4.1: Let k be a positive integer. We define two sequences $\alpha_k(n)$ and $\beta_k(n)$ recursively as follows: $\alpha_k(1) = -k$, $\beta_k(1) = -3$, and

$$\alpha_k(n+1) = \alpha_k(n) + \alpha_k(1) - (n+1),$$
$$\beta_k(n+1) = \beta_k(n) - k + \frac{(n+1)^2}{2} - (n+1)\alpha_k(1) + \beta_k(1).$$

Theorem 4.2: *Let k be a positive integer. Consider a generic surface $\Sigma_n \in |\mathcal{O}_{\mathbb{P}^3}(n)|$. Then for any integer d, and any list W of simple singularity types* satisfying*

$$\mu(S) \le k \quad \text{for any } S \in W, \tag{3}$$

and

$$\sum_{S\in W} \mu(S) \le \frac{nd^2}{2} + \alpha_k(n)d + \beta_k(n), \tag{4}$$

there exists a curve $C \in |\mathcal{O}_{\Sigma_n}(d)|$ having W as its set of singularities. Moreover, $H^1(\Sigma_n, \mathcal{I}_C^{es}(d-1)) = H^1(\Sigma_n, \mathcal{I}_C^{es}(d)) = 0$, where $\mathcal{I}_C^{es}$ denotes the equisingular ideal of the singularities of the curve C.

Remark 4.3:

1. It follows from the definition that
$$\alpha_k(n) = -kn - \frac{n(n+1)}{2},$$
$$\beta_k(n) = -n(3+k) + k\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{12}.$$
2. Theorem 4.2 provides us with an asymptotically optimal result, since $\dim|\mathcal{O}_{\Sigma_n}(d)| = nd^2/2 + O(d)$.

* Any type can appear several times.

Proof: The proof is by induction on n. For $n=1$ the statement was proven in [21]. Assuming that the statement holds for n, let us prove it for $n+1$. To do this we shall degenerate Σ_{n+1} into a union $\mathbb{P}^2 \cup \Sigma_n$ and use patchworking.

Consider a generic surface Σ_n of degree n and a generic plane $\mathbb{P}^2_0$. Choosing an arbitrary smooth surface $\widetilde{\Sigma}_{n+1}$ of degree $n+1$, we construct the pencil $s \cdot (\mathbb{P}^2_0 \cup \Sigma_n) + t \cdot \widetilde{\Sigma}_{n+1}$; $(s:t) \in \mathbb{P}^1$, which provides us with the flat family of surfaces $X \to \mathbb{P}^1$, having $\mathbb{P}^2_0 \cup \Sigma_n$ as its central fibre, and having smooth generic fibre. To obtain the family of line bundles on X we just pull back the sheaf $\mathcal{O}_{\mathbb{P}^3}(d)$.

Next, we shall construct the curve in the central fibre. Consider a partition of $W = W_1 \cup W_2$ satisfying

$$0 \le \frac{nd^2}{2} + \alpha_k(n)d + \beta_k(n) - \sum_{\mathcal{S} \in W_1} \mu(\mathcal{S}) < k.$$

Such a partition exists due to condition (3). It follows (by the definition of $\alpha_k(n)$ and $\beta_k(n)$) that

$$\sum_{\mathcal{S} \in W_2} \mu(\mathcal{S}) \le \frac{(d-n-1)^2}{2} + \alpha_k(1)(d-n-1) + \beta_k(1).$$

Applying the induction assumption to the pair (Σ_n, W_1), we obtain a curve $C_1 \subset |\mathcal{O}_{\Sigma_n}(d)|$ having W_1 as its set of singularities. By generality of $\mathbb{P}^2_0$ we can assume that the intersection $C_1 \cap \mathbb{P}^2_0$ is transversal. Moreover, we can assume that the intersection $C_1 \cap \mathbb{P}^2_0$ is generic.* Next, due to [21], we can choose a curve $C_2 \in |\mathcal{O}_{\mathbb{P}^2_0}(d)|$ having W_2 as its set of singularities and satisfying $H^1(\mathbb{P}^2_0, \mathcal{I}^{es}_{C_2}(d-n-1)) = 0$. Moreover, varying C_2 in its equisingular family and restricting it to $\Sigma_n \cap \mathbb{P}^2_0$, we can obtain any generic element in $|\mathcal{O}_{\Sigma_n \cap \mathbb{P}^2_0}(d)|$ (the proof is exactly the same as before). Hence we can choose C_2 in such a way that $C_2 \cap \Sigma_n = C_1 \cap \mathbb{P}^2_0$, or, in other words, in such a way that the union $C_1 \cup C_2$ is given by a section of $\mathcal{O}_{\Sigma_n \cup \mathbb{P}^2_0}(d)$.

Now, applying the weak patchworking theorem and Theorem 3.1 (twice), we derive the existence of a curve $C \in |\mathcal{O}_{\Sigma_{n+1}}(d)|$ having W as its set of singularities. The h^1-vanishing part follows from the semi-continuity of cohomology, and from the appropriate h^1-vanishing on the reducible surface $\Sigma_n \cup \mathbb{P}^2_0$. ∎

* $H^0(\Sigma_n, \mathcal{I}^{es}_{C_1}(d)) = 0$ is the tangent space to the equisingular deformation of C_1. We know that $H^1(\Sigma_n, \mathcal{I}^{es}_{C_1}(d-1)) = 0$, hence the natural map $H^0(\Sigma_n, \mathcal{I}^{es}_{C_1}(d)) \to H^0(\Sigma_n \cap \mathbb{P}^2_0, \mathcal{O}_{\mathbb{P}^3}(d))$ is surjective, which implies the required statement.

4.2 THE SECOND EXAMPLE. To present the second example we will need the following definition

Definition 4.4: Let $\mathcal{S}$ be a type of plane curve singularity (either topological or analytic), and let $C \subset \mathbb{P}^2$ be an algebraic curve. We say that C is a good representative of $\mathcal{S}$ if

- C has exactly one singular point of type $\mathcal{S}$ as its only singularity,
- the germ $V \subset |\mathcal{O}_{\mathbb{P}^2}(\deg(C))|$ of the equisingular/equianalytic strata of C is smooth of expected dimension (i.e., minimal possible).

Define $s(\mathcal{S})$ to be the minimal integer, such that there exists a good representative C of the singularity $\mathcal{S}$ of degree $\deg(C) = s(\mathcal{S})$.

Assume that we are given a projective algebraic surface Σ and a curve $C \subset \Sigma$ satisfying:

- C has exactly r singular points $z_1, \ldots, z_r$, and those are smooth points of Σ.
- All the singular points of C are ordinary multiple points of the multiplicities $m_1, \ldots, m_r$.
- The germ at C of the equisingular deformation $V_{|\mathcal{O}_\Sigma(C)|}(m_1, \ldots, m_r)$ is smooth and has expected dimension.
- C is a generic element* of the linear system of curves in $|\mathcal{O}_\Sigma(C)|$ passing through $z_1, \ldots, z_r$ with multiplicities $m_1, \ldots, m_r$.

Then the following statement holds.

THEOREM 4.5: *Let $\mathcal{S}^1, \ldots, \mathcal{S}^r$ be topological types of plane curve singularities. Assume that $s(\mathcal{S}^i) < m_i$ for all i. Then there exists a small deformation C_t, $t \in D_\epsilon(0)$ of C such that for any $t \neq 0$, C_t has exactly r singular points of types $\mathcal{S}^1, \ldots, \mathcal{S}^r$ as its only singularities.*

This theorem was first proven in [17] by tedious direct computation. Here we present a geometric proof based on the patchworking techniques.

Proof: We shall use a generalization of the patchworking pattern presented in Example 2.1. Define $X = Bl_{z_1 \times 0, \ldots, z_r \times 0}(\Sigma \times \mathbb{P}^1)$ to be the blow up of the trivial family $\Sigma \times \mathbb{P}^1$ at the points $z_1 \times 0, \ldots, z_r \times 0$. Then X admits natural projections $\pi: X \to \mathbb{P}^1$ and $\alpha: X \to \Sigma$, and the zero fibre of π satisfies $X_0 = \bigcup_{i=0}^r \Sigma^i$, where Σ^0 is the blow up of Σ at $z_1, \ldots, z_r$ and, for any $i > 0$, $\Sigma^i = E_i$ is the exceptional

* This assumption is not necessary, but it simplifies slightly the proof of Theorem 4.5.

divisor corresponding to the point $z_i \times 0$. Define

$$\mathcal{L} = \alpha^* \mathcal{O}_\Sigma(C) \otimes \bigotimes_{i=1}^{r} \mathcal{O}_X(-m_i E_i).$$

Now it is important to mention that if $2\dim|\mathcal{O}_\Sigma(C)| < \sum_{i=1}^r m_i(m_i+1)$, then the dimension $h^0(X_t, \mathcal{L}_t)$ jumps at $t = 0$! Hence one cannot use the weak patchworking theorem. However, the strong version of the patchworking is still applicable in this case.

To complete the patchworking pattern we have to construct a curve $D \in |\mathcal{L}_0|$ satisfying certain properties. As the first step we consider the curve $C_0 = X_0 \cap \overline{C \times (\mathbb{P}^1 - \{0\})}$, and denote by C_0^i the intersection $C_0 \cap \Sigma^i$. Then C_0^0 is the proper transform of C, and C_0^i are collections of lines through some points $q^i \in \Sigma^i$, $i = 1, \ldots, r$. Consider the equisingular (i.e., equimultiple) family $V_{C_0}(m_1, \ldots, m_r) \subset H^0(X_0, \mathcal{L}_0)$. Then C_0 belongs to the intersection $V_{C_0}(m_1, \ldots, m_r) \cap (\pi_* \mathcal{L} \otimes k(0))$.

Claim 4.6: *The intersection $V_{C_0}(m_1, \ldots, m_r) \cap (\pi_* \mathcal{L} \otimes k(0))$ is transversal at C_0.*

Claim 4.7: *Let $\mathcal{S}$ be a plane curve singularity type (either topological or analytic), and let $L \subset \mathbb{P}^2$ be a straight line. Denote $m = s(\mathcal{S}) + 1$. Then, for any set of m distinct points $p_1, \ldots, p_m \in L$, there exists a good representative $D \in |\mathcal{O}_{\mathbb{P}^2}(m)|$ of the singularity type $\mathcal{S}$, satisfying $D \cap L = \bigcup_{i=1}^m p_i$. Moreover, the map $H^0(\mathbb{P}^2, I_D^{es/ea}(m)) \to H^0(L, \mathcal{O}_L(m))$ is surjective.*

We postpone the proofs of the claims and finish first the proof of the theorem. Let $D^1, \ldots, D^r$ be good representatives of the singularity types $\mathcal{S}^1, \ldots, \mathcal{S}^r$, such that $D^i \cap \Sigma^0 = C_0^0 \cap \Sigma^i$ (the existence of such curves follows from Claim 4.7). Thus the curve $D = C_0^0 \cup D^1 \cup \cdots \cup D^r$ belongs to the linear system $|\mathcal{L}_0|$. Consider the irreducible equisingular family $V_D(\mathcal{S}^1, \ldots, \mathcal{S}^r)$ containing the curve D. This family is smooth, and it is invariant under the action of the group $\mathbb{G} = \prod_{i=1}^r \mathbb{G}^i$, where $\mathbb{G}^i$ denotes the group of automorphisms of Σ^i, acting trivially on $\Sigma^0 \cap \Sigma^i$. Hence its closure contains the variety $V_{C_0}(m_1, \ldots, m_r)$. By Claim 4.6 the intersection $V_{C_0}(m_1, \ldots, m_r) \cap (\pi_* \mathcal{L} \otimes k(0))$ is transversal at C_0. Consider a $\mathbb{G}$-equivariant Whitney stratification $\overline{V_D(\mathcal{S}^1, \ldots, \mathcal{S}^r)} = W_0 \supset W_1 \supset \cdots$. We can assume that there exists i such that $W_i \subset V_{C_0}(m_1, \ldots, m_r)$ is dense. Then $C_0 \in W_i$ due to our assumption on generality of C. Thus the intersection $V_D(\mathcal{S}^1, \ldots, \mathcal{S}^r) \cap (\pi_* \mathcal{L} \otimes k(0))$ is transversal at any point $C' \in V_D(\mathcal{S}^1, \ldots, \mathcal{S}^r) \cap (\pi_* \mathcal{L} \otimes k(0))$ sufficiently close to C_0.

Now we can apply the strong patchworking theorem to the patchworking data $(X \to \mathbb{P}^1, \mathcal{L}, D)$. Condition $T1$ is satisfied by the construction of D. Condition $T3$ follows from the transversality of $V_D(S^1, \ldots, S^r) \cap (\pi_* \mathcal{L} \otimes k(0))$ at D. Thus it remains to prove that condition $T2$ is also satisfied. Let $\mathcal{I}$ be the equisingular/equianalytic ideal of the curve D. Consider the exact sequence of sheaves

$$0 \to \mathcal{I} \otimes \mathcal{L}_0 \to \bigoplus_{i \geq 0} \mathcal{I} \otimes (\mathcal{L}_0)_{|_{\Sigma^i}} \to \bigoplus_{i>0} (\mathcal{L}_0)_{|_{\Sigma^i \cap \Sigma^0}} \to 0.$$

By Claim 4.7 the map $\bigoplus_{i \geq 0} H^0(X_0, \mathcal{I} \otimes (\mathcal{L}_0)_{|_{\Sigma^i}}) \to \bigoplus_{i>0} H^0(X_0, (\mathcal{L}_0)_{|_{\Sigma^i \cap \Sigma^0}})$ is surjective. Hence the natural map

$$H^1(X_0, \mathcal{I} \otimes \mathcal{L}_0) \to \bigoplus_{i \geq 0} H^1(X_0, \mathcal{I} \otimes (\mathcal{L}_0)_{|_{\Sigma^i}}) = H^1(\Sigma^0, (\mathcal{L}_0)_{|_{\Sigma^0}})$$

is an isomorphism (the last equality follows from the fact that D^i are good representatives of the corresponding singularity types). Similarly, considering the exact sequence

$$0 \to \mathcal{L}_0 \to \bigoplus_{i \geq 0} (\mathcal{L}_0)_{|_{\Sigma^i}} \to \bigoplus_{i>0} (\mathcal{L}_0)_{|_{\Sigma^i \cap \Sigma^0}} \to 0$$

and its long exact sequence of cohomology we obtain the natural isomorphism $H^1(X_0, \mathcal{L}_0) \to H^1(\Sigma^0, (\mathcal{L}_0)_{|_{\Sigma^0}})$, which implies condition $T2$, namely the natural map $H^1(X_0, \mathcal{I} \otimes \mathcal{L}_0) \to H^1(X_0, \mathcal{L}_0)$ is also an isomorphism. ∎

Proof of Claim 4.6: It is enough to show that

$$H^0(X_0, I^{es}_{C_0} \otimes \mathcal{L}_0) + (\pi_* \mathcal{L} \otimes k(0)) = H^0(X_0, \mathcal{L}_0),$$

which is equivalent to surjectivity of the map $\pi_* \mathcal{L} \otimes k(0) \to H^0(X_0, \mathcal{O}_{Z^{es}(C_0)})$. Consider the scheme $Z = \overline{Z^{es}(C) \times (\mathbb{P}^1 - \{0\})} \subset X$. Then $Z^{es}(C_0) = Z \otimes k(0)$. The dimension of $\mathcal{O}_{Z \otimes k(t)}$ is constant, thus the sheaf $\pi_* \mathcal{O}_Z$ is locally free and the natural map $\pi_* \mathcal{O}_Z \otimes k(t) \to H^0(X_t, \mathcal{O}_{Z \otimes k(t)})$ is an isomorphism. It is clear that the map $\pi_* \mathcal{L} \otimes k(0) \to H^0(X_0, \mathcal{O}_{Z^{es}(C_0)})$ factors through $\pi_* \mathcal{O}_Z \otimes k(0)$. Thus it is enough to prove that the map $\pi_* \mathcal{L} \otimes k(t) \to \pi_* \mathcal{O}_Z \otimes k(t)$ is surjective for all t.

For $t \neq 0$, this is equivalent to the assumption that the germ at C of the equisingular (i.e., equimultiple) strata is smooth and has expected dimension. Now one can easily derive the surjectivity for $t = 0$ by a straightforward computation (in local coordinates near the points z_i) using the fact that Z is a trivial family of zero-dimensional schemes. We leave this computation to the reader. ∎

Proof of Claim 4.7: We will prove the claim using Theorem 2.8. Let $z \in \mathbb{P}^2$ be a generic point. Define X to be the blow up of $\mathbb{P}^2 \times \mathbb{P}^1$ along the point $z \times 0$, and along the lines $p_i \times \mathbb{P}^1$, $i = 1, \dots, m$. Then X admits natural projections $\sigma: X \to \mathbb{P}^2$ and $\pi: X \to \mathbb{P}^1$. Denote the exceptional divisors corresponding to the lines $p_i \times \mathbb{P}^1$ by E^i, and that corresponding to the point $z \times 0$ by E^0. We should mention that the zero fibre X_0 of our family X is the union of the plane E^0 with the blow up $\widetilde{\mathbb{P}^2}$ of $\mathbb{P}^2 \times 0$ at $z, p_1, \dots, p_m$.

The second ingredient of the patchworking pattern is the line bundle $\mathcal{L}$, which is given by

$$\mathcal{L} = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(m)) \otimes \mathcal{O}_X(-(m-1)E^0) \otimes \bigotimes_{i=1}^{r} \mathcal{O}_X(-E^i).$$

To complete the patchworking pattern we shall find the curve $C_0 \in |\mathcal{L}_0|$. Since $s(\mathcal{S}) = m - 1$ we can choose a good representative $C_0^0 \in |\mathcal{L}_{|_{E^0}}|$ of the singularity type $\mathcal{S}$. Without loss of generality we assume that $C_0^0 \cap \widetilde{\mathbb{P}^2}$ consists of $m - 1$ distinct points, such that none of these points belongs to any straight line connecting E^0 with E^i, $i = 1, \dots, m$. Thus one can extend C_0^0 up to a curve $C_0 \in |\mathcal{L}_0|$, by a smooth curve $C_0^1 \subset \widetilde{\mathbb{P}^2}$.

It is easy to see that the patchworking pattern constructed above satisfies all the properties $X1, X2, S1, \dots, S5$. So, to finish the proof, it remains to verify (1). We shall do this using Theorem 3.1. It is enough to prove that

$$H^1(E^0, \mathcal{I}(m-1)) = 0 \quad \text{and} \quad H^1(\widetilde{\mathbb{P}^2}, \mathcal{L} \otimes \mathcal{O}(-E^0 \cap \widetilde{\mathbb{P}^2})) = 0.$$

The first equality holds, since C_0^0 is a good representative of $\mathcal{S}$. To prove the second equality we observe that

$$H^1(\widetilde{\mathbb{P}^2}, \mathcal{L} \otimes \mathcal{O}(-E^0 \cap \widetilde{\mathbb{P}^2})) = H^1(\mathbb{P}^2, \mathcal{J}(m-1)),$$

where $\mathcal{J}$ is the sheaf of ideals of the zero-dimensional scheme of fat points $z^{m-1} \cup p_1 \cup \cdots \cup p_m$. We show, by induction on m, that the last group is zero.

For $m = 1$ the statement is obvious. To make the induction step we shall consider the exact sequence

$$0 \to \mathcal{J}_1(m-2) \to \mathcal{J}(m-1) \to \mathcal{O}_L(-1) \to 0,$$

where $\mathcal{J}_1$ is the sheaf of ideals of the zero-dimensional scheme of fat points $z^{m-2} \cup p_1 \cup \cdots \cup p_{m-1}$, and L is the line passing through z and p_m. Now, the required h^1-vanishing follows from the Riemann–Roch theorem because of the induction assumption.

To summarize: we proved that for any t, small enough, there exists an irreducible reduced curve $C_t \in |\mathcal{L}_t|$ having exactly one singular point of type $\mathcal{S}$ as its only singularity. Consider $\sigma(C_t) \subset \mathbb{P}^2$. It is a curve of degree m passing through $p_1, \ldots, p_m$ and having exactly one singular point of type $\mathcal{S}$ as its only singularity. $\sigma(C_t)$ is a small deformation of C_0, thus it is a good representative of $\mathcal{S}$.

To prove the 'moreover' part, it is sufficient to show that

$$H^1(\mathbb{P}^2, I_{C_t}^{es/ea}(m-1)) = 0$$

for t small enough. C_0^0 is a good representative of the singularity type $\mathcal{S}$, thus $H^1(\mathbb{P}^2, I_{C_0^0}^{es/ea}(m-1)) = 0$. Hence $H^1(\mathbb{P}^2, I_{C_t}^{es/ea}(m-1)) = 0$ by the semicontinuity of the cohomology. ∎

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